

# On geodesic propagators and black hole holography

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## Abstract

One of the most challenging technical aspects of the dualities between string theory on anti-de Sitter spaces and conformal field theories is understanding how location in the interior of spacetime is represented in the field theory. It has recently been argued that the interior of the spacetime can be directly probed by using intrinsically non-local quantities in the field theory. In addition, Balasubramanian and Ross [hep-th/9906226] argued that when the spacetime described the formation of an  $\text{AdS}_3$  black hole, the propagator in the field theory probed the whole spacetime, including the region behind the horizon. We use the same approach to study the propagator for the BTZ black hole and a black hole solution with a single exterior region, and show that it reproduces the propagator associated with the natural vacuum states on these spacetimes. We compare our result with a toy model of the CFT for the single-exterior black hole, finding remarkable agreement. The spacetimes studied in this work are analytic, which makes them quite special. We also discuss the interpretation of this propagator in more general spacetimes, shedding light on certain issues involving causality, black hole horizons, and products of local operators on the boundary.

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# 1 Introduction

The proposed duality between string theory on anti-de Sitter space and lower-dimensional conformal field theory [1] provides a non-perturbative definition of string theory, and could thus, subject to the restriction on the asymptotic boundary conditions, cast a bright light on many dark corners of quantum gravity. In particular, the field theory description encompasses arbitrary fluctuations of the metric and other fields in the interior, and should provide a fully quantum description of the formation and evaporation of a black hole. One of the major barriers to studying conceptual questions in quantum gravity using this theory is our poor understanding of how an approximately local classical (or semi-classical) spacetime description of the physics emerges from the fundamental gauge theory description, and the consequent absence of any intuition about how this approximate locality breaks down under extreme conditions. (A related problem is that in the regime where a classical spacetime description is a good approximation, we don't have any other quantitative description; see [2] for a recent attempt to construct computationally useful approximations.)

The connection between asymptotic behavior of the spacetime fields and the field theory was one of the first subjects of study [3,4], and it was subsequently shown that the map between states in the field theory and states in spacetime identifies the asymptotic behavior of the fields with the expectation values of local operators in the field theory [5]. This was used to show a “scale-radius duality” for a variety of bulk sources, and for wavepackets of supergravity fields – the radial position of a bulk probe is encoded in the scale size of the dual expectation values. Dynamical sources for supergravity fields were studied in [6], where the radial position of a source particle following a bulk geodesic was reflected in the size and shape of an expectation-value bubble in the CFT. The expectation values of the operators produced by spacetime sources were further studied in [7–13].

However, the simple scale-radius relationship seen in these studies is a consequence of an isometry in pure AdS space which is dual to a scale transformation in the conformal field theory, under which the vacuum remains invariant. For situations describing black holes, which break the symmetries, the relationship between bulk position and boundary observables will be more complicated [6, 14]. The same phenomenon is apparent in the collision of two massless particles to form a black hole in [13]; after the particles collide, their radial position is fixed, but the scales in the boundary expectation values continue to evolve.

Furthermore, the asymptotic values of the fields are not sufficient to reproduce the whole spacetime. Since asymptotic values of fields in AdS space are dual to the expectation values of local operators in the CFT, it follows that such expectation values describe only a small piece of the physical information. A number of authors have studied spacetime sources which do not change the asymptotic values of the fields, such as particles

in  $\text{AdS}_3$  and spherical shells, and found that the location of the shell or particles is encoded in non-local operator expectation values in the field theory, such as the two-point function and Wilson loops [15–18]. Similar work is described in [19]. Thus, non-local operators must be included in any understanding of the bulk-boundary connection. A particularly striking case is asymptotically  $\text{AdS}_3$  spaces, where we can describe a wide range of dynamics without relying on perturbations around some background [20, 21], and the asymptotic metric only encodes the total mass and angular momentum of the system.

In [15], Balasubramanian and Ross used a stationary phase approximation to obtain predictions for the propagator in the gauge theory from the geodesics of supergravity solutions in which a black hole was formed. This propagator appeared to be sensitive to events in the interior of the black hole. Now, while the CFT may well encode information about the black hole interior, the particular CFT propagator studied in [15] is in fact the restriction to the boundary of  $\text{AdS}$  space of a propagator associated with the bulk quantum field theory. This raises certain issues about causality<sup>1</sup> which we wish to clarify in the work below. Some general arguments are presented in section 2. In short, we argue that the propagator studied in [15] is in fact a causal object, but that the stationary phase approximation is valid only in appropriately analytic spacetimes and not in the actual spacetime considered in [15]. However, even without the stationary-phase approximation, the path-integral definition of the propagator used in [15] should generally lead to a result which depends on the region inside the black hole; we argue that this should be interpreted as an object which is defined by a mixture of past and future boundary conditions.

We then proceed to explore the propagator in two analytic spacetimes in order to see more precisely what sort of object it represents. The spacetimes that we consider contain black holes, but are static outside the Killing horizon. In those cases, the stationary phase approximation is expected to be valid, and a computation of the propagator reduces to a study of various geodesics in the bulk spacetime. We show that, in such cases, the propagator of [15] is in fact the boundary limit of a time-ordered expectation value of a product of local bulk fields. Our spacetimes are the spinless BTZ black hole [22] and the associated  $\mathbb{RP}^2$  geon [23]. We find that the propagator in each case is associated with a natural vacuum state for linearized quantum fields on the spacetime, and that geodesics passing behind the black hole horizon play an important role in determining the structure of this state. The states are analogues of the Hartle-Hawking state, and are defined by boundary conditions at past and future infinity. The propagators in these cases are known to be Green’s functions of a (causal) wave equation, and sensitivity to ‘events’ behind the event horizon would once again seem to contradict this causality. In

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<sup>1</sup>These issues were brought to our attention by Lenny Susskind through his comments at the Val Morin workshop on Black Holes, June 1999.

this case, the resolution is that the analyticity of these spacetimes implies that much of the information about the region inside the event horizon is in fact ‘stored outside’. Note however that knowledge of the region outside the Killing horizon is not enough to determine what happens inside the (future) event horizon; we also need access to the ‘white hole’ region, inside the past event horizon.

The next section is devoted to a short commentary on the AdS/CFT correspondence and to general arguments concerning the nature of the calculations in [15]. Section 3 then reviews the BTZ and geon spacetimes and determines the propagators on these spacetimes given by the path integral of [15]. In section 4, these calculations are compared to the propagator in the dual CFT. We discuss the extension of the propagator calculation to the rotating BTZ black hole spacetime in an appendix.

## 2 The setting and the approximations

We use this section to set the stage for our later calculations. The most relevant elements of the AdS/CFT correspondence are briefly reviewed in section 2.1. This allows us to discuss the particular regime in which we use the correspondence and to comment on certain subtleties. We then address the stationary phase approximation and the issue of causality in section 2.2. Section 2.3 includes a few further comments on the interpretation of the propagator.

### 2.1 The correspondence in the bulk classical limit

While the AdS/CFT correspondence is conjectured to relate the full quantum theories associated with bulk string theory and the CFT, it is fair to say that this correspondence is best understood in the neighborhood of the vacuum. In that region, a useful way to describe the correspondence is in terms of the partition functions  $Z_{CFT}$  and  $Z_{bulk}$ , which in both cases are functions of external sources that may be coupled to the CFT and to the boundary of the AdS space. Recall that the CFT lives on a spacetime which may be identified with the boundary of  $AdS_3$ . The partition functions are equal and, by differentiating them, we may arrive at relations between propagators and correlators in the two theories. For example, differentiating twice yields the relation [7]

$$\langle \mathcal{O}_\partial(b), \mathcal{O}_\partial(b') \rangle_\partial = \lim_{\epsilon \rightarrow 0} \epsilon^{-2\Delta} \langle \mathcal{O}_B(b_\epsilon) \mathcal{O}_B(b'_\epsilon) \rangle_B \quad (1)$$

between the propagators in the boundary and bulk, where the bulk operators  $\mathcal{O}_B$  are at points  $b_\epsilon, b'_\epsilon$  in the bulk that approach the points  $b, b'$  in a certain way as  $\epsilon \rightarrow 0$ . (also see [3, 4]). This is a relation between the Euclidean propagators or, via analytic continuation, between the Feynman propagators in the respective vacuum states. Since

we are in the vacuum state, operators on the right-hand side may be viewed as fields on AdS space.

In the work below, we again wish to consider a propagator or correlator. However, we wish to work in a regime that is rather far from the vacuum state. We consider a state in which the bulk string theory is nearly classical and contains, or is in the process of forming, a large black hole. Since the bulk string theory is nearly classical, quantum fluctuations are infinitesimal and are well approximated by linear fields. In terms of the CFT, this is the limit of large 't Hooft coupling. While this is not the classical limit of the CFT, it is a limit in which we again expect certain kinds of classical behavior (such as factorization of correlation functions with infinitesimal corrections).

Now, by acting on the vacuum with a sufficient set of local operators, we should be able to reach any state in the Hilbert space. Thus, the relation between the partition functions implies that any state  $|\Phi\rangle_{\partial}$  in the CFT will be associated with some state  $|\Phi\rangle_B$  in the bulk. Unfortunately, it is difficult to describe this relationship in detail. Nonetheless, given any bulk state and an associated state in the CFT, it follows that correlation functions in the CFT state are given much as above by the limit of correlation functions in the bulk as the points are moved to the boundary of the spacetime.

As stated above, the regime of interest here is the limit in which the bulk spacetime is nearly classical and in which the quantum fluctuations are effectively linear. This is just the usual setting of (free) quantum field theory in curved spacetime. As a result, it is clear that a given classical geometry does not determine a unique quantum state, but rather determines an entire space of states for the linearized fluctuations. For globally static spacetimes, one can identify a preferred vacuum state, though this is not generally possible. For example, in the familiar asymptotically flat black hole spacetimes, the ‘natural’ choices of state for the linear quantum fields include the Hartle-Hawking vacuum as well as the Unruh vacuum, and more complicated choices of state are possible as well. The particular choice of quantum state may be associated with initial and/or final conditions satisfied by the linearized fluctuations.

Now [15] used the relation (1) to link a CFT object to a bulk propagator. As a result, some particular choice of state, or perhaps several states or a class of states, for the linearized bulk quantum fields must have been made implicitly. We note that in [15] it was explicitly assumed that the ‘propagator’ for a scalar field  $\phi$  in the bulk was given by the path integral expression

$$\langle\phi(x)\phi(x')\rangle_{\text{FPI}} = \int d\mathcal{P} e^{i\Delta L(\mathcal{P})}, \quad (2)$$

where  $L(\mathcal{P})$  denotes the length of the path  $\mathcal{P}$ . The measure  $d\mathcal{P}$  was not specified in detail as the intention of [15] was to use the expression (2) only in the semiclassical approximation. The subscript FPI reminds us that this is the object defined by a Feynman path integral, to distinguish it from other two-point functions that we may

wish to discuss. The conventions are set here so that spacelike paths have positive imaginary length, while timelike paths have real length. The question we wish to explore is whether this is in fact the 2-point function of any bulk quantum state and, if so, just which state it represents.

Now, the two-point function alone does not uniquely determine the quantum state. However, for linear fields there is the notion of a quasi-free state (see, e.g., [24]), also known as a Gaussian state, in which the higher connected  $n$ -point functions vanish, and all of the structure is in fact determined by the two-point function. It is therefore natural to attempt to associate the calculations of [15] with a quasi-free state of the linearized bulk fields. We will show below that, on the BTZ black hole spacetime, the expression (2) does in fact yield the 2-point function of the Hartle-Hawking vacuum state. Similarly, on the  $\mathbb{RP}^2$  geon spacetime, it yields the 2-point function of the so called geon-vacuum, the analogue of the state discussed in [25] for asymptotically flat geons. These states are in fact quasi-free. In subsection 2.3 below, we will discuss to what extent we can draw the same conclusion in more general spacetimes.

## 2.2 Causality and the stationary phase approximation

After calculating the propagator (2) using a stationary phase method, it was found in [15] that this propagator was sensitive to events happening behind a black hole's event horizon. This raises certain issues about causality. Stated most simply, we have noted (see eq. 1) that the correlation functions in the CFT are (up to a rescaling) the boundary limits of correlation functions in the bulk. However, in the current context of bulk correlators for linear quantum field theory in curved spacetime, it is well known that the evolution is causal. An operator at any point in the spacetime can be expressed purely in terms of operators in its past light cone. How, therefore, are we to interpret the results of [15] which suggest that correlation functions of such operators near the boundary are sensitive to the interior of the black hole?

In order to address this question, we first provide a few words on the general interpretation of the propagator (2). Let us first note that there are at least two natural ways that we might try and interpret this object. The first is as a (time-ordered) correlation function in some quantum state. For definiteness, let us use the word 'state' in the sense of algebraic quantum field theory. This means that a 'state'  $\rho$  may be either a pure state or a mixed state and that we would try to interpret (2) as  $Tr(\rho T(\phi(x)\phi(y)))$  for some  $\rho$ . The second natural choice is to try to interpret the propagator as the time-ordered version of a transition amplitude:  $\langle\alpha|T(\phi(x)\phi(y))|\beta\rangle$ . In either case, however, the propagator would be a Green's function for the wave operator and thus a causal object.

Thus, we need to know whether the propagator (2) does in fact yield a Green's function for the wave operator  $\nabla^2$ . That this is the case may be argued as follows. Let

us consider the spacetime as the configuration space of a “non-relativistic particle” and take  $H = \nabla^2$  to be its Hamiltonian<sup>2</sup>. As usual, we may write

$$\frac{-i}{H} = \int_0^\infty e^{-iN(H-i\epsilon)} dN, \quad (3)$$

so that the object on the right hand side defines a Green’s function for the wave operator. By the usual path integral skeletonization arguments, one can write this as

$$\langle x | \frac{-i}{H} | y \rangle = \int_0^\infty dN \int DxDp \exp i \int_0^1 [\dot{x}p - N(p^2 + m^2)] d\lambda, \quad (4)$$

where  $\dot{\phantom{x}} = d/d\lambda$ . We will see in a moment that (4) is just the path integral (2) in another form. Alternatively, (4) could be taken as the definition of the path integral (2) [26–28].

The path integral above contains the action for a free relativistic particle. Note, however, that while such particles are typically associated with a time reparametrization invariance, there is no such explicit invariance above. We may thus consider (4) to be a gauge-fixed path integral, using in particular the gauge  $\dot{N} = 0$ :

$$\langle x | \frac{-i}{H} | y \rangle = \int_0^\infty dN \int DxDp \delta(\dot{N}) \exp i \int_0^1 [\dot{x}p - N(p^2 + m^2)] d\lambda. \quad (5)$$

The argument below will be more transparent if we change the gauge fixing scheme to use a gauge condition that depends only on the path  $x(\lambda)$  through position space<sup>3</sup>. Thus, we write

$$\langle x | \frac{-i}{H} | y \rangle = \int_0^\infty dN \int DxDp \Delta(x) \exp i \int_0^1 [\dot{x}p - N(p^2 + m^2)] d\lambda \quad (6)$$

where  $\Delta(x)$  contains both the gauge fixing condition and the associated Faddeev-Popov determinant. Note that  $\Delta(x)$  will depend only on  $x(\lambda)$ .

Now, to lowest order in the WKB approximation, performing an integral over some variable is equivalent to solving the associated classical equation of motion and inserting the result back into the action. Thus, we can do the integrals over  $N$  and  $p$  and write the result as follows:

$$\langle x | \frac{-i}{H} | y \rangle = \int Dx \Delta'(x) \exp iL(x(\lambda)), \quad (7)$$

where  $L(x(\lambda))$  is the length of the path  $x(\lambda)$  with exactly the same conventions as in (2).

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<sup>2</sup>Which, in this case, is unbounded from below due to the Lorentzian signature of the spacetime.

<sup>3</sup>A complete such gauge fixing cannot be a smooth function of the path  $x(\lambda)$ , but this need not concern us here.

The factor  $\Delta'(x)$  denotes  $\Delta(x)$  together with the various path-dependent measure factors arising from the corrections to the WKB approximation in integrating over  $N$  and  $p$ . Identifying  $d\mathcal{P} = \Delta'(x)Dx$ , we find that our Green's function is just the propagator (2). Note that solving the equation of motion for  $N$  involves taking a square root. For the timelike segments of path the restriction  $N > 0$  was used to choose the appropriate branch. For the spacelike segments, the appropriate branch is determined by the details of the measure as discussed in [28]. Note that the action is an analytic function of both  $N$  and  $p$  so that we expect no problems with the use of stationary phase methods here. Thus, the propagator (2) is indeed a Green's function for a wave operator. That (2) satisfies Dirichlet boundary conditions on the smooth part of the boundary at infinity can be seen from the arguments of [29].

At this point, we can now reduce our physical question about causality in the setting of [15] to a mathematical question about solutions of the wave equation. In [15] further stationary phase methods were used to argue that, to leading order, the propagator was in fact determined by the shortest geodesic connecting the points  $x$  and  $y$ . The authors considered a spacetime that was pure AdS before a certain spacelike hypersurface  $\Sigma$  on which two massless point particles entered through the boundary at infinity. From this it is clear that two points sufficiently far in the past of  $\Sigma$  can only be connected by geodesics that lie in the pure AdS part of the spacetime. Thus, the geodesic approximation leads to the conclusion that, to the past of some hypersurface  $\Sigma'$ , the propagator is just as it would be in pure AdS space<sup>4</sup>.

Nonetheless, at sufficiently late times, it was shown in [15] that there are points outside the black hole such that the shortest geodesic connecting them runs through the interior of the black hole. It was therefore concluded that the propagator (2) outside the black hole was sensitive to events occurring inside the black hole.

In order to eliminate certain technical worries, let us consider a family of generalizations of the spacetime constructed in [15]. Imagine replacing the singular null particles with a distribution of null fluid of compact support. Since there is no local gravitational dynamics in 2+1 dimensions, the resulting spacetime is easily made identical to that of [15] outside of the region occupied by the null fluid. Until the formation of the black hole singularity, the resulting spacetime is then smooth<sup>5</sup>. If the field  $\phi$  for which we compute the propagator does not couple to the null fluid, then the definition of the propagator on this spacetime remains just (2). Thus, we have a complete specification of the propagator, up to issues associated with the black hole singularity<sup>6</sup>.

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<sup>4</sup>In that case, as we will discuss below, it is known to be the time ordered 2-point function in the AdS vacuum.

<sup>5</sup>It is not, however, asymptotically AdS where the null fluid enters the spacetime. We shall assume that this does not cause any further complications.

<sup>6</sup>Such issues certainly exist. For example, if we take (4) as the definition of (2), the black hole singularity will imply that  $H$  is not essentially self-adjoint and that some particular self-adjoint extension



Suppose now that we arrange things such that the two bits of null fluid actually collide inside the black hole. That is, suppose that at some event the supports of the two distributions of fluid overlap. Note that, depending on the sort of null fluid used, various outcomes are possible. Some sorts of fluid would interpenetrate readily while other sorts would bounce solidly off of each other. The outcome should affect some of the geodesics mentioned above that connect two points near infinity by passing through the interior of the black hole.

Now we see that we have a real contradiction at hand. On the one hand, we have the statement that the propagator at early times is the AdS vacuum correlator – independent of what goes on in the black hole interior. Also, we know that the propagator satisfies the wave equation and so evolves in a causal fashion. Thus, the propagator at points outside the black hole can be expressed in terms of initial data on an early hypersurface in a manner that is independent of what goes on in the black hole interior. Thus, the propagator outside the black hole cannot in fact depend on events inside the black hole. This is in direct contradiction to the conclusion of the previous paragraph.

The resolution seems to be that the geodesic ‘approximation’ is not in fact a valid approximation<sup>7</sup>. In retrospect, it seems quite likely that this approximation fails for such a spacetime. Note that to arrive at the geodesic approximation, one would use a stationary phase argument to solve the classical equations of motion corresponding to the action  $m \int \sqrt{-\dot{x}^2}$ . While the stationary point (the spacelike geodesic) does indeed lie on the original contour of integration (real values of  $x$ ), this contour is not a steepest descent contour through the stationary point. In particular, in a Lorentzian signature spacetime, a spacelike geodesic is not a path of minimal length. As a result, if one wishes to argue that the stationary point dominates, one must first analytically continue the action to complex values of the coordinates and attempt to deform the original contour to the contour of steepest descent.

Now, the action involves the metric  $g_{ab}(x)$ . To avoid the issue of the singularities, let us consider the smoothed spacetimes with null fluid sources. Since the fluid density vanishes in an open region, but not in the entire spacetime, it is clear that such spacetimes are not analytic and that continuation is problematic. Thus, it is not at all clear that steepest descent methods should succeed in this case, and we are happy to associate their failure with nonanalyticities of the spacetime.

While this seems to settle the issue nicely, we should mention for completeness that, if one excises the region of non-zero fluid density from the spacetime, the resulting spacetime does have a real analytic atlas and can be continued. Presumably, excising the region occupied by the fluid prevents one from deforming the contours as one would

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should be chosen. Here, we simply assume that some such choice has been made.

<sup>7</sup>It is also a logical possibility that the approximation is valid, but simply unstable in a manner that causes higher order effects at early times to evolve into lower order effects at late times.

like.

## 2.3 Interpreting the Propagator

Having ruled out the use of the geodesic approximation in general, what are we to conclude about the full propagator (2)? In principle, picking any two points  $x$  and  $y$  in the spacetime, the path integral includes contributions from paths connecting them that explore arbitrarily far into the future. As a result, even in the spacetime studied in [15], it is far from clear that the propagator at early times is independent of events in the interior of the black hole. It seems likely that the propagator does not correspond to a fixed initial condition, but instead to some mixture of initial and final conditions. In this case, the propagator between points near the boundary at late times may depend on events inside the black hole as well. That is, it may be possible to choose a state or states in such a way that the two-point function reproduces the important qualitative features found in the calculation in [15]. Such a state (or states) will involve a mixture of initial and final conditions, reflecting the fact that the form of the two-point function in [15] depends on the formation of the black hole in the future.

Let us return to the two natural interpretations of the propagator mentioned above: as a time ordered expectation value in some quasi-free state, and as a time ordered transition amplitude between two states. We note that either is compatible with the above observations. In the case of the expectation value, it may simply be the case that the quantum state itself is one that is naturally defined by a combination of retarded and advanced boundary conditions, and so is free to depend on events in the interior of the black hole. We note that the Hartle-Hawking state for an asymptotically flat black hole is an example of such a state that is naturally associated with boundary conditions in both past and future, while the Unruh state is associated only with boundary conditions in the past. In the case of the transition amplitude, both states may involve such ‘mixed’ boundary conditions, or perhaps one is defined by retarded boundary conditions and one by advanced boundary conditions.

In spacetimes that are asymptotically flat at both timelike and spacelike infinity, the propagator (2) can be shown to define a transition amplitude [30]. On the other hand, the work of Wald [31] effectively shows that (2) defines an expectation value for globally static spacetimes (without horizons). It is also known to give the expectation value of time-ordered fields in the Hartle-Hawking state on the Kruskal spacetime, though the status of this question on a general black hole spacetime is not yet understood [24]. We will see that an expectation value is once again obtained on the spinless BTZ spacetime and the associated  $\mathbb{RP}^2$  geon.

### 3 The geodesics in $\text{AdS}_3$ and quotient spacetimes

We have argued in section 2 that stationary phase methods do not in general yield a valid approximation to the FPI propagator (2). Nevertheless, one may ask if there are cases in which it does provide a valid approximation and, if so, whether geodesics passing behind the horizon play any important role. We shall see in this section and the next that the answer to both of these questions is in the affirmative.

In the present section, we consider the lengths of spacelike geodesics in the  $\text{AdS}_3$ , spinless BTZ, and  $\mathbb{RP}^2$  geon spacetimes. As these spacetimes are real Lorentzian sections of holomorphic complex manifolds, one may expect the geodesic approximation to succeed in these cases. Indeed, it is known [15] to succeed in yielding the vacuum correlator on  $\text{AdS}_3$ . In the following section, we consider the propagators obtained through this approximation, and compare to what we know about the field theory. This will allow us to explicitly check the agreement with certain CFT calculations and to trace the role of geodesics passing through the interior of the black hole. The final agreement provides additional confirmation of the accuracy of the geodesic approximation in these cases.

In fact, these calculations are not truly independent. Since the spinless BTZ and  $\mathbb{RP}^2$  geon spacetimes are quotients of  $\text{AdS}_3$ , a method of images argument together with analytic continuations and the uniqueness of the Euclidean Green's functions shows that the success of the geodesic approximation to (2) in reproducing the vacuum correlator on  $\text{AdS}_3$  implies that it must also approximate the Hartle-Hawking correlation function for the spinless BTZ hole and the related geon correlation function (see [25]) on the  $\mathbb{RP}^2$  geon. Thus, in these cases the FPI propagator gives the expectation value of a time-ordered product of fields in a quasi-free state.

#### 3.1 Geodesics of $\text{AdS}_3$

The  $\text{AdS}_3$  spacetime can be constructed as the hyperboloid

$$(T^1)^2 + (T^2)^2 - (X^1)^2 - (X^2)^2 = 1 \quad (8)$$

in a flat embedding space with metric

$$ds^2 = -(dT^1)^2 - (dT^2)^2 + (dX^1)^2 + (dX^2)^2. \quad (9)$$

Here, we are choosing units so that the AdS length scale  $\ell$  (related to the cosmological constant) is one. A set of intrinsic coordinates on  $\text{AdS}_3$  is given in terms of these embedding coordinates by

$$T^1 = \cosh \chi \cos \tau, T^2 = \cosh \chi \sin \tau, X^1 = \sinh \chi \sin \varphi, X^2 = \sinh \chi \cos \varphi, \quad (10)$$

where  $\varphi$  has period  $2\pi$ , and  $0 \leq \chi \leq \infty$ . For the hyperboloid,  $\tau$  is also periodic with period  $2\pi$ , but we pass to the covering space, and take  $\tau$  to run between  $\pm\infty$ . In terms of these coordinates, the metric is

$$ds^2 = d\chi^2 + \sinh^2 \chi d\varphi^2 - \cosh^2 \chi d\tau^2 = \left( \frac{2}{1 - \rho^2} \right)^2 (d\rho^2 + \rho^2 d\varphi^2) - \left( \frac{1 + \rho^2}{1 - \rho^2} \right)^2 d\tau^2. \quad (11)$$

In the second equality, we have defined a new radial coordinate  $\rho = \tanh(\chi/2)$ , so  $0 \leq \rho \leq 1$ . Fixed  $\tau$  surfaces have the Poincaré disc geometry, and the dual CFT is defined on a cylinder isomorphic to the  $\rho = 1$  boundary.

We will need the length of the unique geodesic traveling between  $(\tau, \chi_m, \pm\varphi_m)$ . Now, since the metric at fixed  $\tau$  is that of the Poincaré disc, equal-time geodesics of (11) are circle segments obeying the equation

$$\tanh \chi \cos(\varphi - \alpha) = \cos(\beta), \quad (12)$$

where the geodesic reaches the  $\chi = \infty$  boundary at  $\varphi = \alpha \pm \beta$ . Setting  $\alpha = 0$ , the unique geodesic between the boundary points  $(\tau, \pm\beta)$  intersects  $\chi = \chi_m$  at  $\varphi_m^\pm$  which are fixed by

$$\tanh \chi_m \cos \varphi_m^\pm = \cos(\pm\beta), \quad (13)$$

which implies that

$$-\varphi_m^- = \varphi_m^+ \equiv \varphi_m. \quad (14)$$

Integrating (11) yields the length of the geodesic connecting  $(\tau, \chi_m, \pm\varphi_m)$ :

$$L(\varphi_m, -\varphi_m) = 2 \ln \left[ \sinh \chi_m \sin \varphi_m + (\sinh^2 \chi_m \sin^2 \varphi_m + 1)^{1/2} \right]. \quad (15)$$

### 3.2 Spacelike geodesics on the spinless BTZ hole

The spinless BTZ hole is obtained by taking the quotient of the region  $T^1 > |X^1|$  of  $\text{AdS}_3$  by the isometry  $\exp(2\pi r_+ \xi)$ , where  $\xi$  is the Killing vector

$$\xi = X^1 \frac{\partial}{\partial T^1} + T^1 \frac{\partial}{\partial X^1}. \quad (16)$$

To express this geometry in the Schwarzschild-like coordinates of the original papers [22], we introduce on the region  $X^2 > |T^2|$ ,  $T^1 > 0$  of  $\text{AdS}_3$  the coordinates  $(t, r, \phi)$  by

$$\begin{aligned} T^1 &= \frac{r}{r_+} \cosh(r_+ \phi), \\ X^1 &= \frac{r}{r_+} \sinh(r_+ \phi), \end{aligned}$$

$$\begin{aligned}
T^2 &= \left( \frac{r^2}{r_+^2} - 1 \right)^{1/2} \sinh(r_+ t), \\
X^2 &= \left( \frac{r^2}{r_+^2} - 1 \right)^{1/2} \cosh(r_+ t).
\end{aligned} \tag{17}$$

$t$  and  $\phi$  take all real values,  $r > r_+$ , and the metric takes the form

$$ds^2 = -N^2 dt^2 + r^2 d\phi^2 + \frac{1}{N^2} dr^2; \quad N^2 = r^2 - 8GM, \tag{18}$$

where  $M = r_+^2/(8G)$ . The identification by  $\exp(2\pi r_+ \xi)$  amounts to  $(t, r, \phi) \sim (t, r, \phi + 2\pi)$ , and with this identification the coordinates  $(t, r, \phi)$  cover one exterior region of the BTZ hole.

We are interested in geodesics between two points,  $x_1$  and  $x_2$ , in the exterior region of the hole. We take the value of  $r$  at both points to be the same. To parametrize the locations of the points, let  $y_1$  and  $y_2$  be two points in  $\text{AdS}_3$ , respectively at  $(t_1, r, \phi_1)$  and  $(t_2, r, \phi_2)$ , and let  $x_1$  (respectively  $x_2$ ) be the equivalence class of  $y_1$  ( $y_2$ ). We write  $\Delta\phi = \phi_2 - \phi_1$  and  $\Delta t = t_2 - t_1$ , and we assume that  $|\Delta\phi + 2\pi n| > |\Delta t|$  for all integers  $n$ . For fixed  $t_1, t_2, \phi_1$ , and  $\phi_2$ , it is then straightforward to show that for sufficiently large  $r$  there are countably many spacelike geodesics connecting  $x_1$  and  $x_2$  in the BTZ hole.

To calculate the lengths of these geodesics, we exploit the symmetries to argue that the geodesic distance between  $y_1$  and  $y_2$  in  $\text{AdS}_3$  is a function only of the chordal distance  $D$  in the embedding space,

$$\begin{aligned}
D &= -(\Delta T^1)^2 - (\Delta T^2)^2 + (\Delta X^1)^2 + (\Delta X^2)^2 \\
&= \frac{4r^2}{r_+^2} \sinh^2 \left( \frac{r_+ \Delta\phi}{2} \right) - 4 \left( \frac{r^2}{r_+^2} - 1 \right) \sinh^2 \left( \frac{r_+ \Delta t}{2} \right).
\end{aligned} \tag{19}$$

By considering a simple example of a spacelike geodesic, we can show that the relation between chordal distance and proper length  $L$  is

$$\sinh^2(L/2) = \frac{D}{4}. \tag{20}$$

It then follows from the quotient construction that the lengths,  $L_n(x_1, x_2)$ , of the geodesics connecting  $x_1$  and  $x_2$  in the BTZ hole have the large  $r$  expansion

$$\exp[L_n(x_1, x_2)] = \frac{2r^2}{r_+^2} \{ \cosh[r_+(\Delta\phi + 2\pi n)] - \cosh(r_+ \Delta t) \} + O(1), \tag{21}$$

where  $n \in \mathbb{Z}$ .

### 3.3 Spacelike geodesics on the $\mathbb{RP}^2$ geon

Recall [23] that the  $\mathbb{RP}^2$  geon is obtained by taking the quotient of the region  $T^1 > |X^1|$  of  $\text{AdS}_3$  by the isometry that is the composition of  $J_1 : \exp(\pi r_+ \xi)$  and the involution  $J_2 : (T^1, T^2, X^1, X^2) \mapsto (T^1, T^2, X^1, -X^2)$ . The resulting spacetime is not orientable, but one can construct a related orientable spacetime from the product of the BTZ spacetime with  $T^4$ . If the moduli of the  $T^4$  are chosen so that there is an orientation-reversing involution  $J_4$  of the torus, then one obtains an orientable spacetime by taking the quotient with respect to  $J_1 \circ J_2 \circ J_4$ .

Now, let  $y_1$  and  $y_2$  be points on  $\text{AdS}_3$  as above, respectively at  $(t_1, r, \phi_1)$  and  $(t_2, r, \phi_2)$ , and suppose that  $|\Delta\phi + 2\pi n| > |\Delta t|$  for all integers  $n$ . Let  $x_1$  and  $x_2$  be two points in the exterior region of the geon, such that  $x_1$  (respectively  $x_2$ ) is the equivalence class of  $y_1$  ( $y_2$ ). For sufficiently large  $r$ , one class of spacelike geodesics connecting  $x_1$  and  $x_2$  is then obtained precisely as for the BTZ hole, with the result (21) for their lengths. The second class of geodesics arises from the  $\text{AdS}_3$  geodesics connecting  $y_1$  to the points  $\tilde{y}_{2;n}$ , located at

$$\begin{aligned} T^1 &= (r/r_+) \cosh[r_+(\phi_2 + \pi + 2\pi n)] \ , \\ X^1 &= (r/r_+) \sinh[r_+(\phi_2 + \pi + 2\pi n)] \ , \\ T^2 &= \sqrt{(r/r_+)^2 - 1} \sinh(r_+ t_2) \ , \\ X^2 &= -\sqrt{(r/r_+)^2 - 1} \cosh(r_+ t_2) \ , \end{aligned} \tag{22}$$

where  $n \in \mathbb{Z}$ . As

$$\begin{aligned} D(y_1, \tilde{y}_{2;n}) &= 2(r/r_+)^2 \{ \cosh[r_+(\phi_2 - \phi_1 + \pi + 2\pi n)] - 1 \} \\ &\quad + 2[(r/r_+)^2 - 1] \{ \cosh[r_+(t_2 + t_1)] + 1 \} \ , \end{aligned} \tag{23}$$

the lengths  $\tilde{L}_n(x_1, x_2)$  of these geodesics have the large  $r$  expansion

$$\exp \left[ \tilde{L}_n(x_1, x_2) \right] = \frac{2r^2}{r_+^2} \{ \cosh[r_+(\phi_2 - \phi_1 + \pi + 2\pi n)] + \cosh[r_+(t_2 + t_1)] \} + O(1) . \tag{24}$$

It is precisely this class of geodesics that pass through the black hole interior. We note that all such geodesics are *longer* than the shortest geodesic connecting  $x_1$  and  $x_2$  through the exterior region. Thus, at first sight one might think that geodesics passing through the interior cannot be relevant to leading order. Nonetheless, we shall see in section 4.3 that they do provide the leading contribution to the two-particle correlations in the geon vacuum, and that (24) reproduces expectations based on the dual CFT.

## 4 Matching to the CFT

It turns out that, due to difficulties in performing the various mode sums, there are few exact results for the bulk correlators in the spinless BTZ Hartle-Hawking state and in the geon vacuum. We will therefore proceed by comparing the limiting behaviors of (30) and (24) with expectations based on toy models of the dual CFT. We shall see that the agreement is surprisingly good. This supports both the accuracy of the bulk geodesic approximation in these cases and the ability of the toy models to capture much of the physics of the CFT. We first review the calculation showing that the geodesic approximation in  $\text{AdS}_3$  reproduces the vacuum propagator, and then show that the asymptotic behavior of (24) reproduces the expected two-particle correlations in BTZ and the geon.

### 4.1 The propagator in $\text{AdS}_3$

We will now review the calculation of the equal time correlation functions in the dual field theory for the  $\text{AdS}_3$  geometry using the (bulk) WKB approximation. A scalar field of mass  $m$  in a spacetime which is asymptotically  $\text{AdS}_3$  is dual to an operator  $\mathcal{O}$  of conformal dimension  $\Delta = 1 + \sqrt{1 + m^2}$ . The fiducial metric for the CFT on the cylinder is related to the induced metric obtained from (11) by a diverging Weyl factor. To relate operators to expectation values, we need to regulate this behavior by cutting off the spacetime at a boundary defined by

$$\rho_m(\tau, \varphi) = 1 - \epsilon(\tau, \varphi), \quad \epsilon(\tau, \varphi) = \epsilon(\tau, -\varphi), \quad (25)$$

where  $\epsilon$  is some smooth function of the boundary coordinates. The symmetry of  $\epsilon$  under  $\varphi \rightarrow -\varphi$  is chosen for simplicity. For the calculations relating to the BTZ black hole and geon, we will take the cutoff surface to be at constant  $r$  in the BTZ coordinates. According to [7], the Feynman propagator for  $\mathcal{O}$  in the dual CFT is obtained from the spacetime propagator between the corresponding points on the cutoff boundary at  $\rho_m$  (also see [3, 4]),

$$G_\partial((\tau, \varphi), (\tau', \varphi')) = \epsilon^{-2\Delta} G_B(\mathbf{B}, \mathbf{B}'), \quad (26)$$

where  $\mathbf{B} = (\tau, \varphi, \rho_m(\tau, \varphi))$ . We will only need the propagator when  $\tau = \tau'$ . For  $\mathbf{B}, \mathbf{B}'$  causally unrelated, the Green's function  $G_B(\mathbf{B}, \mathbf{B}')$  in the leading order semi-classical approximation is given by a sum over geodesics:

$$G(\mathbf{B}, \mathbf{B}') = \sum_g e^{-\Delta L_g(\mathbf{B}, \mathbf{B}')}. \quad (27)$$

Here  $L_g$  is the (real) geodesic length between the boundary points and only spacelike geodesics contribute since  $\tau = \tau'$ .

By rotational invariance, it is sufficient to perform the calculation for  $\varphi = -\varphi'$ . For the particular case  $1 - \epsilon(\tau, \varphi) = \tanh(\chi_m/2) = \text{const}$ , the length of the geodesic connecting  $\mathbf{B}$  and  $\mathbf{B}'$  is given by (15). In fact, the symmetry  $\epsilon(\tau, \varphi) = \epsilon(\tau, -\varphi)$  guarantees that a corresponding result holds for any such symmetric choice of  $\epsilon$ . So, to leading order in  $\epsilon$ , the geodesic length between the points  $\mathbf{B}, \mathbf{B}'$  is

$$L(\mathbf{B}, \mathbf{B}') = 2 \ln \left( \frac{2 \sin \varphi}{\epsilon} \right). \quad (28)$$

The bulk propagator is thus

$$G(\mathbf{B}, \mathbf{B}') = \left( \frac{2 \sin \varphi}{\epsilon} \right)^{-2\Delta} \quad (29)$$

in the  $\epsilon \rightarrow 0$  limit, where the boundary metric is  $ds^2 = (1/\epsilon(\tau, \varphi)^2)(-d\tau^2 + d\varphi^2)$ . This correctly reproduces the CFT two-point correlator of [3] for  $\Delta\tau = 0$  and  $\Delta\varphi = 2\varphi$ , since the CFT is defined on the Weyl rescaled cylinder with metric  $ds^2 = -d\tau^2 + d\varphi^2$ .

## 4.2 The propagator in BTZ

We now apply the bulk geodesic approximation method of [15] to the Green's function on the boundary of the spinless BTZ hole, using the geodesic length (21). The geodesic approximation to the path integral (2) reads

$$\begin{aligned} \langle \phi(x_1) \phi(x_2) \rangle_{\text{FPI}} &= \int d\mathcal{P} e^{i\Delta L(\mathcal{P})} \approx \sum_n \exp[-\Delta L_n(x_1, x_2)] \\ &= \left( \frac{r_+^2}{2r^2} \right)^\Delta \sum_{n=-\infty}^{\infty} \frac{1}{\{\cosh[r_+(\Delta\phi + 2\pi n)] - \cosh(r_+\Delta t)\}^\Delta} \\ &\quad + O\left( \left( \frac{r_+^2}{r^2} \right)^{\Delta+1} \right). \end{aligned} \quad (30)$$

This is the bulk propagator; to relate it to the boundary propagator, we observe that at large  $r$ , the boundary metric in the BTZ spacetime is  $ds^2 = r^2(-dt^2 + d\phi^2)$ . Since we want the CFT to live on the same cylinder as above, the boundary propagator is given by  $G_\partial = r^{2\Delta} G_B$  in this case. The rescaling thus precisely cancels the  $r^{-2\Delta}$  in the prefactor.

We note that this Green's function is manifestly periodic in the global time  $t$  in the imaginary direction, and the period  $2\pi/r_+$  is the inverse of the spinless BTZ temperature. It is thus plausibly identified with the propagator in the analogue of the Hartle-Hawking state for this black hole. We also note that the geodesics used in this calculation lie



entirely outside the black hole. This calculation successfully reproduces the result given in [8] for the spinless BTZ black hole. A similar agreement is obtained for the rotating BTZ black hole in appendix A.

### 4.3 The propagator in the single-exterior black hole

We now proceed to address the dual CFT propagator associated with the bulk FPI propagator  $\langle x_1 x_2 \rangle_{\text{geon}}$  on the  $\mathbb{RP}^2$  geon. Now, the corresponding path integral can be written as a sum of two contributions:

$$\langle x_1 x_2 \rangle_{\text{geon}} = \langle x_1 x_2 \rangle_{\text{BTZ}} + \langle x_1 J(x_2) \rangle_{\text{BTZ}} \quad (31)$$

where  $\langle x_1 x_2 \rangle_{\text{BTZ}}$  represents the bulk FPI propagator on the spinless BTZ hole. Here, we take  $x_1$  and  $x_2$  to lie outside the geon horizon so that we may naturally associate them with two points in an asymptotic region of the BTZ black hole. The first term ( $\langle x_1 x_2 \rangle_{\text{BTZ}}$ ) was calculated in the geodesic approximation in section 3.2 while the second term ( $\langle x_1 J(x_2) \rangle_{\text{BTZ}}$ ) is given in the geodesic approximation by (24). The geodesics that contribute to this second term are longer than the shortest geodesic contributing to  $\langle x_1 x_2 \rangle_{\text{BTZ}}$ , so that one might at first think that  $\langle x_1 J(x_2) \rangle_{\text{BTZ}}$  can be neglected. However, let us now Fourier transform this result in order to compute the two-particle correlations in the geon state. Since the energies of the two particles cannot add to zero, the time translation invariance of the BTZ hole is enough to guarantee that the contributions from the first term (with both points in the same asymptotic region) vanish. However, the contribution of the second term need not vanish, corresponding to the fact that the geon does not itself have a time translation invariance. Thus, we see that the two-particle correlations in the geon state can be directly tied to the second term above, which results only from geodesics that pass through the interior of the BTZ black hole. In terms of the geon spacetime, the result is again that only geodesics passing behind the horizon can account for the two-particle correlations.

Thus, we might try to match these correlations to those computed in [23] for a toy model of the CFT state  $|\text{geon}\rangle$  dual to the  $\mathbb{RP}^2$  geon, presumably with linearized quantum fluctuations in the geon vacuum state. The toy model replaced the CFT by a free scalar field and found, in the case of a nontwisted field, the correlations

$$\langle \text{geon} | d_{n,\epsilon} d_{n',\epsilon'} | \text{geon} \rangle = \frac{(-1)^n \delta_{n,n'} \delta_{\epsilon,-\epsilon'}}{2 \sinh(\pi n / r_+)} \quad , \quad (32)$$

where  $d_{n,\epsilon}$  is the annihilation operator for the mode with frequency quantum number  $n$ ,  $n = 1, 2, \dots$ , and the index  $\epsilon$  takes the value 1 for right-movers and  $-1$  for left-movers (see (33) below). Here and below we ignore issues involving the zero mode ( $n = 0$ ). As a consequence of rotational invariance, the correlations are between a right-mover and

a left-mover with the same frequency. We note that this nontwisted free scalar field has conformal weight  $\Delta = 0$ .

We now show that this result can be obtained from the geodesic approximation (24) to the bulk Green's function. It is clear, however, that due to the simplified nature of the toy model, one should not expect to be able to Fourier transform the asymptotic values of the propagator and obtain (32) directly. In particular, the bulk propagator will not be built from only the discrete mode spectrum of (32). In the full interacting CFT, the correlator will similarly not be periodic in time, and so will not be a simple combination of these discrete modes. However, we may attempt to extract information analogous to (32) by modifying the Fourier transform of (24) to take into account the fact that the bulk propagator is not periodic in time. We shall see that the agreement is impressive.

In the toy (free) CFT, the oscillator modes that correspond to the annihilation operators  $d_{n,\epsilon}$  in (32) are [23]

$$u_{n,\epsilon} = \frac{1}{\sqrt{4\pi n}} e^{-in(t-\epsilon\phi)} \quad (33)$$

where  $n = 1, 2, \dots$  and  $\epsilon = \pm 1$ . If the Green's function  $G(t_1, \phi_1; t_2, \phi_2)$  had the periodicity of the oscillator modes, we would thus have

$$\begin{aligned} \langle d_{n,\epsilon} d_{n',\epsilon'} \rangle &= \frac{\sqrt{nn'}}{4\pi^3} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \\ &\times \exp[i(n t_1 + n' t_2 - n\epsilon\phi_1 - n'\epsilon'\phi_2)] G(t_1, \phi_1; t_2, \phi_2). \end{aligned} \quad (34)$$

We shall modify (34) to take into account the lack of periodicity shortly.

As discussed above, the part of  $G(t_1, \phi_1; t_2, \phi_2)$  coming from the geodesics that do not pass through the geon does not contribute to  $\langle d_{n,\epsilon} d_{n',\epsilon'} \rangle$ . The part of  $G(t_1, \phi_1; t_2, \phi_2)$  coming from the geodesics that do pass through the geon is, from (24),<sup>8</sup>

$$\left(\frac{r_+^2}{2}\right)^\Delta \sum_{k=-\infty}^{\infty} \frac{1}{\{\cosh[r_+(\phi_2 - \phi_1 + \pi + 2\pi k)] + \cosh[r_+(t_2 + t_1)]\}^\Delta} . \quad (35)$$

As (35) depends on  $\phi_1$  and  $\phi_2$  only through the combination  $\phi_2 - \phi_1$ , integrating over  $\phi_2 + \phi_1$  in (34) is immediate. Next, we observe that each term in (35) depends on  $\phi_2 - \phi_1$  and  $k$  only through the combination  $\phi_2 - \phi_1 + 2\pi k$ . Integrating  $\phi_2 - \phi_1$  from zero to  $2\pi$  and summing over  $k$  is thus equivalent to integrating any one term in (35) in  $\phi_2 - \phi_1$  from negative infinity to positive infinity. Writing the integration in terms of the variable

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<sup>8</sup>The asymptotic metric in the geon spacetime is the same as in the BTZ spacetime, so the rescaling relating the bulk and boundary propagators is the same as in the previous subsection.

$y := \epsilon(\phi_2 - \phi_1 + \pi)$ , we obtain

$$\begin{aligned} \langle d_{n,\epsilon} d_{n',\epsilon'} \rangle &= \frac{\sqrt{nn'}}{2\pi^2} \delta_{n\epsilon, -n'\epsilon'} (-1)^n \left( \frac{r_+^2}{2} \right)^\Delta \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \\ &\quad \times \int_{-\infty}^{\infty} dy \frac{\exp[i(nt_1 + n't_2 + ny)]}{\{\cosh(r_+ y) + \cosh[r_+(t_2 + t_1)]\}^\Delta}. \end{aligned} \quad (36)$$

We must now face the fact that the integrand in (36) is not periodic in  $t_1$  and  $t_2$ . We reinterpret (36) by hand so that  $t_2 + t_1 := \alpha$  is integrated over  $\mathbb{R}$  but  $t_2 - t_1$  over  $4\pi$ . The integral over  $t_2 - t_1$ , combined with the Jacobian that arises from the change of variables, yields then  $2\pi\delta_{n,n'}$ . The factor  $\delta_{n\epsilon, -n'\epsilon'}$  can thus be replaced by  $\delta_{\epsilon, -\epsilon'}$ , and we find

$$\begin{aligned} \langle d_{n,\epsilon} d_{n',\epsilon'} \rangle &= \frac{n}{\pi} \delta_{n,n'} \delta_{\epsilon, -\epsilon'} (-1)^n \left( \frac{r_+^2}{2} \right)^\Delta \\ &\quad \times \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} dy \frac{\exp[in(\alpha + y)]}{[\cosh(r_+ \alpha) + \cosh(r_+ y)]^\Delta}. \end{aligned} \quad (37)$$

Changing variables to  $u = \alpha - y$ ,  $v = \alpha + y$ , gives finally [32]

$$\begin{aligned} \langle d_{n,\epsilon} d_{n',\epsilon'} \rangle &= \frac{n}{2\pi} \delta_{n,n'} \delta_{\epsilon, -\epsilon'} (-1)^n \left( \frac{r_+}{2} \right)^{2\Delta} \int_{-\infty}^{\infty} \frac{du}{[\cosh(r_+ u/2)]^\Delta} \int_{-\infty}^{\infty} \frac{dv \exp(inv)}{[\cosh(r_+ v/2)]^\Delta} \\ &= \frac{n}{2\pi} \delta_{n,n'} \delta_{\epsilon, -\epsilon'} (-1)^n (r_+)^{2(\Delta-1)} \left( \frac{\Gamma(\Delta/2)}{\Gamma(\Delta)} \right)^2 \Gamma\left(\frac{\Delta}{2} + \frac{in}{r_+}\right) \Gamma\left(\frac{\Delta}{2} - \frac{in}{r_+}\right). \end{aligned} \quad (38)$$

In the limit  $\Delta \rightarrow 0_+$ , (38) reduces to

$$\begin{aligned} \langle d_{n,\epsilon} d_{n',\epsilon'} \rangle &= \frac{2n}{\pi} \delta_{n,n'} \delta_{\epsilon, -\epsilon'} (-1)^n (r_+)^{-2} \Gamma\left(\frac{in}{r_+}\right) \Gamma\left(-\frac{in}{r_+}\right) \\ &= \frac{4}{r_+} \times \frac{(-1)^n \delta_{n,n'} \delta_{\epsilon, -\epsilon'}}{2 \sinh(\pi n/r_+)} , \end{aligned} \quad (39)$$

which agrees with (32) up to the factor  $4/r_+$ . This factor may be a consequence of our having neglected any pre-exponential factors in the bulk Green's function, or from our by-hand reinterpretation of the  $dt_1 dt_2$  integrals in (34).

This result verifies the importance of geodesics passing behind the horizon in obtaining the proper 2-particle correlations, and shows that the toy free CFT does indeed match well with the bulk spacetime results. As discussed earlier, it is only in special spacetimes which are appropriately analytic than we can expect the geodesic approximation to hold. As a result, the fact that our calculation relies on geodesics passing behind

the horizon of the black hole is consistent with the causal nature of the FPI propagator and with the ideas of [13] that one must look beyond simple products of local operators in the CFT to encode useful information about the interior of a black hole.

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## A The propagator for the rotating BTZ hole

In this appendix we generalize the treatment of sections 3.2, 4.2 to show that the bulk geodesic approximation method of [15] reproduces the Green's function in the Poincaré vacuum (see [8] and the references therein) on a single boundary component of the rotating nonextremal BTZ hole.

The generalization of equations (17) to the rotating case is the rotating exterior BTZ coordinate transformation [22]

$$\begin{aligned} T^1 &= \sqrt{\alpha} \cosh(r_+ \phi - r_- t), \\ X^1 &= \sqrt{\alpha} \sinh(r_+ \phi - r_- t), \\ T^2 &= \sqrt{\alpha - 1} \sinh(r_+ t - r_- \phi), \\ X^2 &= \sqrt{\alpha - 1} \cosh(r_+ t - r_- \phi), \end{aligned} \tag{40}$$

with

$$\alpha = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}, \tag{41}$$

where  $r > r_+$ ,  $-\infty < t < \infty$ , and  $-\infty < \phi < \infty$ , and the parameters  $r_{\pm}$  satisfy  $0 \leq r_- < r_+$ . For  $r_- = 0$ , this transformation reduces to the spinless transformation (17).

Introducing the points  $y_1$  and  $y_2$  in  $\text{AdS}_3$  as in section 3.2, respectively at  $(t_1, r, \phi_1)$  and  $(t_2, r, \phi_2)$ , we find

$$\begin{aligned} D(y_1, y_2) &= 2\alpha [\cosh(r_+ \Delta \phi - r_- \Delta t) - 1] \\ &\quad - 2(\alpha - 1) [\cosh(r_+ \Delta t - r_- \Delta \phi) - 1] . \end{aligned} \tag{42}$$

When  $t_1, t_2, \phi_1$ , and  $\phi_2$  are fixed, and such that  $|\Delta\phi| > |\Delta t|$ , equation (42) shows that  $D(y_1, y_2) > 0$  for sufficiently large  $r$ .  $y_1$  and  $y_2$  can then be joined by a spacelike geodesic, and the length  $L(y_1, y_2)$  of this geodesic has the large  $r$  expansion

$$\exp [L(y_1, y_2)] = \frac{2r^2}{r_+^2} [\cosh(r_+\Delta\phi - r_-\Delta t) - \cosh(r_+\Delta t - r_-\Delta\phi)] + O(1) \quad . \quad (43)$$

Now, in the region of  $\text{AdS}_3$  covered by the exterior BTZ coordinates, the rotating BTZ quotient construction amounts to the identification  $(t, r, \phi) \sim (t, r, \phi + 2\pi)$ . Let again  $x_1$  (respectively  $x_2$ ) be the equivalence class of the point  $y_1$  ( $y_2$ ). Assuming  $|\Delta\phi + 2\pi n| > |\Delta t|$  for all integers  $n$ , and proceeding as in section 4.2, we find that the geodesic approximation to the path integral (2) reads

$$\begin{aligned} \langle \phi(x_1)\phi(x_2) \rangle_{\text{FPI}} &= \int d\mathcal{P} e^{i\Delta L(\mathcal{P})} \approx \sum_n \exp [-\Delta L_n(x_1, x_2)] \\ &= \left( \frac{r_+^2}{2r^2} \right)^\Delta \sum_{n=-\infty}^{\infty} \frac{1}{\left\{ \cosh [r_+(\Delta\phi + 2\pi n) - r_-\Delta t] - \cosh [r_+\Delta t - r_-(\Delta\phi + 2\pi n)] \right\}^\Delta} \\ &\quad + O \left( \left( \frac{r_+^2}{r^2} \right)^{\Delta+1} \right) \quad . \end{aligned} \quad (44)$$

The boundary-dependent factor in the leading term in (44) at  $r \rightarrow \infty$  can be rewritten as

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left\{ \sinh \left[ \frac{1}{2}(r_+ - r_-)(\Delta\phi + \Delta t + 2\pi n) \right] \sinh \left[ \frac{1}{2}(r_+ + r_-)(\Delta\phi - \Delta t + 2\pi n) \right] \right\}^\Delta} \quad , \quad (45)$$

which is recognized as the dominant factor in the Green's function in the Poincaré vacuum on the boundary of the rotating BTZ hole (see [8] and the references therein).

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